OPTIMIZATION OF THE BEHAVIOR OF DISTRIBUTED SYSTEMS WITH RANDOM PROPERTIES

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The construction of a time-optimal control of distributed systems with random properties is considered for two problems.

In Problem 1 the distributed system is described by a collection of n integral relations in the presence of a constraint imposed on the norm of the control $u(z, \tau)$ in the space $L_2[\nu_z \times (0 \leqslant \tau \leqslant T)]$. In Problem 2 we study the construction of a time-optimal control of the angular motions and torsional oscillations of an idealized model of an elastic aircraft of the "flying wing" type [1, 2]. In the flight of this model in a homogeneous turbulent atmosphere an inequality constraint is imposed on the energy needed for the creation of the control $u(x, t)$.

1. Formulation of Problem 1. Let the control object be described by the following integral relations:

$$q_i(x, t) = \int_{\nu_z} \int_0^t u(z, \tau) G_i(z, \tau, x, t)\, dz\, d\tau \qquad (i = 1, 2, \ldots, n) \tag{1.1}$$

Here $x = (x_1, x_2, \ldots, x_m)$ and $z = (z_1, z_2, \ldots, z_m)$ are two different points of a region ν in which the process being considered takes place. The region ν is denoted as ν_z when we integrate with respect to z. Symbols t, τ denote distinct instants of time, $q_i(x, t)$ $(i = 1, 2, \ldots, n)$ are functions characterizing the state of the control object, $u(z, \tau)$ is a deterministic control function, $G_i(z, \tau, x, t)$ $(i = 1, 2, \ldots, n)$ are given real random functions. Below we assume that the functions $G_i(z, \tau, x, t)$ $(i = 1, 2, \ldots, n)$ can be represented in the form of the following canonic expansions [3]:

$$G_i(z, \tau, x, t) = G_{i0}(z, \tau, x, t) + \sum_{r=1}^{p} h_r G_{ir}(z, \tau, x, t) \qquad (i = 1, 2, \ldots, n) \tag{1.2}$$

Here the expressions $G_{i0}(z, \tau, x, t) \equiv \langle G_i(z, \tau, x, t)\rangle$ $(i = 1, 2, \ldots, n)$ describe the means of the random functions $G_i(z, \tau, x, t)$, and $G_{ir}(z, \tau, x, t)$ $(i = 1, 2, \ldots, n;\ r = 1, 2, \ldots, p)$ are coordinate functions; h_r $(r = 1, 2, \ldots, p)$ are uncorrelated random variables with zero means and known variances. We assume that

$$G_{ir}(z, \tau, x, t) \in L_2[\nu_z \times (0 \leqslant \tau \leqslant T) \times \nu_x \times (0 \leqslant t \leqslant T)]$$
$$(i = 1, 2, \ldots, n;\ r = 1, 2, \ldots, p)$$

Using expressions (1.2) we can find that the means $\langle q_i(x, t)\rangle$ $(i = 1, 2, \ldots, n)$ and the cross-correlation moments $R_{ik}(x, t)$ $(i, k = 1, 2, \ldots, n)$ of the state functions $q_i(x, t)$ $(i = 1, 2, \ldots, n)$ for some fixed instant t have the following form:

$$\langle q_i(x, t)\rangle = \int_{\nu_z} \int_0^t u(z, \tau) G_{i0}(z, \tau, x, t)\, dz\, d\tau \qquad (i = 1, 2, \ldots, n) \tag{1.3}$$

$$R_{ik}(x, t) = \sum_{r=1}^{p}\left\{\theta_r \int_{\nu_z}\int_0^t \int_{\nu_y}\int_0^t [u(z, \tau) G_{ir}(z, \tau, x, t) u(y, \varphi) G_{kr}(y, \varphi, x, t)] \times \right.$$
$$\left. \times\, dz\, d\tau\, dy\, d\varphi\right\} \qquad (i, k = 1, 2, \ldots, n) \qquad (1.4)$$

Here $\theta_r = \langle h_r^2 \rangle$ $(r = 1, 2, \ldots, p)$ are the variances of the uncorrelated random variables h_r $(r = 1, 2, \ldots, p)$. We introduce the following functions:

$$Q_i(x, t) = \eta_i \langle q_i(x, t)\rangle + \mu_i \langle q_i(x, t)\rangle^2 + \sum_{k=1}^{n} \lambda_{ik} R_{ik}(x, t) \equiv$$
$$\equiv \eta_i \int_{\nu_z}\int_0^t u(z, \tau) G_{i0}(z, \tau, x, t)\, dz\, d\tau + \qquad (i = 1, 2, \ldots, n) \qquad (1.5)$$
$$+ \int_{\nu_z}\int_0^t \int_{\nu_y}\int_0^t u(z, \tau) u(y, \varphi) H_i(z, \tau, y, \varphi, x, t)\, dz\, d\tau\, dy\, d\varphi$$

$$H_i(z, \tau, y, \varphi, x, t) = H_i(y, \varphi, z, \tau, x, t) = \mu_i G_{i0}(z, \tau, x, t) G_{i0}(y, \varphi, x, t) +$$
$$+ \sum_{r=1}^{p}\sum_{k=1}^{n} \theta_r \lambda_{ik} G_{ir}(z, \tau, x, t) G_{kr}(y, \varphi, x, t) \qquad (i = 1, 2, \ldots, n) \qquad (1.6)$$

Here η_i, $\lambda_{ik} = \lambda_{ki}$, μ_i $(i, k = 1, 2, \ldots, n)$ are given weight constants. The functions $Q_i(x, t)$ $(i = 1, 2, \ldots, n)$ can be looked upon as generalized state functions of (1.1). They yield a certain average probability characteristic of the process at the instant t.

Suppose that the condition

$$Q_i(x, T) = \gamma_i(x, T) \qquad (i = 1, 2, \ldots, n) \qquad (1.7)$$

$$\gamma_i(x, T) = \eta_i \int_{\nu_z}\int_0^T u(z, \tau) G_{i0}(z, \tau, x, t)\, dz\, d\tau + \qquad (i = 1, 2, \ldots, n)$$
$$+ \int_{\nu_z}\int_0^T \int_{\nu_y}\int_0^T u(z, \tau) u(y, \varphi) H_i(z, \tau, y, \varphi, x, T)\, dz\, d\tau\, dy\, d\varphi \qquad (1.8)$$

is fulfilled by assumption at some fixed instant $t = T$. Here $\gamma_i(x, T)$ $(i = 1, 2, \ldots, n)$ are given functions satisfying the conditions

$$\gamma_i(x, T) \in L_2[\nu], \qquad \int_{\nu_x} \sum_{i=1}^{n} \gamma_i^2(x, T)\, dx > 0 \qquad (i = 1, 2, \ldots, n) \qquad (1.9)$$

We pose the following problem: find a deterministic control

$$u(z, \tau) \in L_2[\nu_z \times (0 \leqslant \tau \leqslant T)]$$
$$\|u\| \equiv \left(\int_{\nu_z}\int_0^T u^2(z, \tau)\, dz\, d\tau\right)^{1/2} \leqslant a \equiv \text{const} > 0 \qquad (1.10)$$

which takes system (1.1) from the state

$$q_i(x, 0) \equiv 0 \qquad (i = 1, 2, \ldots, n) \tag{1.11}$$

to the state $Q_i(x, T) = \gamma_i(x, T)$ of (1.8) $(i = 1, 2, \ldots, n)$ in the least time T possible. We assume that: (a) at least one solution of system (1.8) exists, (b) the norm $\|u\|$ of the optimal control depends continuously on T.

Note 1. If $h_r = 0$ $(r = 1, 2, \ldots, p)$, then the process being investigated is not stochastic but deterministic. In this case the original problem is a problem of reaching terminal states in least time T under a norm-bounded control. Similar problems for a finite or a countable set of moment equalities of type (1.8), when the moments γ_i $(i = 1, 2, \ldots, n, \ldots)$ do not depend on the space coordinates, have been investigated, for example, in [4, 5]. To solve the original problem we require the results for an auxiliary problem investigated below.

2. Formulation and solution of the auxiliary problem. We pose the following problem: for a fixed instant $t = T$ find an optimal control $u(x, t) \in L_2$ which takes system (1.1) from state (1.11) to the state $Q_i(x, T) = \gamma_i(x, T)$ of (1.8) $(i = 1, 2, \ldots, n)$ with the smallest value of the control norm $\|u\|$.

To solve this problem we apply the Lagrange multiplier method. We set up the auxiliary functional

$$\Xi = -\int\limits_{v_x}\Big\{\sum_{i=1}^{n}\Psi_i(x)\Big[\eta_i\int\limits_{v_z}\int\limits_0^T u(z,\tau)G_{i0}(z,\tau,x,T)\,dz\,d\tau +$$

$$+\int\limits_{v_z}\int\limits_0^T\int\limits_{v_y}\int\limits_0^T u(z,\tau)u(y,\varphi)H_i(z,\tau,y,\varphi,x,T)\,dz\,d\tau\,dy\,d\varphi - \gamma_i(x,T)\Big]\Big\}dx +$$

$$+\int\limits_{v_z}\int\limits_0^T u^2(z,\tau)\,dz\,d\tau \tag{2.1}$$

Here $\Psi_i(x) \in L_2[v]$ $(i = 1, 2, \ldots, n)$ are the Lagrange multipliers. The increment of functional Ξ can be described by the formula

$$\Delta\Xi = \Xi(u, \Delta u) - \Xi(u) = \Pi(u, \Delta u) + \Gamma(\Delta u) \tag{2.2}$$

Here

$$\Pi(u, \Delta u) = \int\limits_{v_z}\int\limits_0^T \Delta u(z,\tau)\Big\{2u(z,\tau) - \Big[\int\limits_{v_x}\sum_{i=1}^{n}\Psi_i(x)\times$$

$$\times\Big(\eta_i G_{i0}(z,\tau,x,T) + 2\int\limits_{v_y}\int\limits_0^T u(y,\varphi)H_i(z,\tau,y,\varphi,x,T)\,dy\,d\varphi\Big)dx\Big]\Big\}dz\,d\tau \tag{2.3}$$

$$\Gamma(\Delta u) = \int\limits_{v_z}\int\limits_0^T \Delta u^2(z,\tau)\,dz\,d\tau - \int\limits_{v_x}\Big\{\sum_{i=1}^{n}\Psi_i(x)\times$$

$$\times\int\limits_{v_z}\int\limits_0^T\int\limits_{v_y}\int\limits_0^T \Delta u(z,\tau)\Delta u(y,\varphi)H_i(z,\tau,y,\varphi,x,T)\,dz\,d\tau\,dy\,d\varphi\Big\}dx \tag{2.4}$$

We can show that the following estimate holds:

$$|\Gamma(\Delta u)| \leqslant \Big\{1 + \Big[\int\limits_{\nu_z}\int\limits_0^T \int\limits_{\nu_y}\int\limits_0^T \Big(\sum_{i=1}^n \Psi_i(x) H_i(z, \tau, y, \varphi, x, T)\, dx\Big)^2 \times$$

$$\times\, dz\, d\tau\, dy\, d\varphi\Big]^{1/2}\Big\} \|\Delta u\|^2 \tag{2.5}$$

It is obvious that $\Pi(u, \Delta u)$ is a functional, additive relative to Δu, for which an estimate such as

$$|\Pi(u, \Delta u)| \leqslant \Big\{2\|u\| + \Big\{\int\limits_{\nu_z}\int\limits_0^T \Big[\int\limits_{\nu_x} \sum_{i=1}^n \eta_i \Psi_i(x) G_{i0}(z, \tau, x, T)\, dx\Big]^2 dz\, d\tau\Big\}^{1/2} +$$

$$+ 2\Big\{\int\limits_{\nu_z}\int\limits_0^T \int\limits_{\nu_y}\int\limits_0^T \Big[\int\limits_{\nu_x} \sum_{i=1}^n \Psi_i(x) H_i(z, \tau, y, \varphi, x, T)\, dx\Big]^2 dz\, d\tau\, dy\, d\varphi\Big\}^{1/2}\Big\} \|\Delta u\| \tag{2.6}$$

turns out to be true. From this we can conclude that $\Pi(u, \Delta u)$ is a linear continuous functional. Taking formula (2.5) into account we can assert that $\Pi(u, \Delta u)$ is a first variation of functional (2.1) [6, 7]. By equating the first variation of functional (2.1) to zero, for determining the optimal control for the auxiliary problem we obtain the integral equation

$$u(z, \tau) - \int\limits_{\nu_y}\int\limits_0^T u(y, \varphi) \int\limits_{\nu_x} \sum_{i=1}^n \Psi_i(x) H_i(z, \tau, y, \varphi, x, T)\, dx\, dy\, d\varphi -$$

$$- \frac{1}{2} \int\limits_{\nu_x} \sum_{i=1}^n \eta_i \Psi_i(x) G_{i0}(z, \tau, x, T)\, dx = 0 \tag{2.7}$$

If we multiply this equation by $u(z, \tau)$ and integrate over the region ν_z and with respect to the parameter τ in the limits from 0 to T, we find the formula

$$\|u\|^2 = \int\limits_{\nu_x} \sum_{i=1}^n \Psi_i(x) \gamma_i(x, T)\, dx - \frac{1}{2} \int\limits_{\nu_x} \Big[\sum_{i=1}^n \Psi_i(x) \times$$

$$\times \int\limits_{\nu_z}\int\limits_0^T u(z, \tau)\, \eta_i G_{i0}(z, \tau, x, T)\, dz\, d\tau\Big] dx \tag{2.8}$$

for computing the smallest value of the control norm.

System (1.8) and Eq. (2.7) form a closed system of $(n + 1)$ equations for determining the $(n + 1)$ unknown functions $u(z, \tau)$, $\Psi_i(x)$ $(i = 1, 2, \ldots, n)$. If we substitute expression (1.6) into formula (2.7), then after using the notation

$$g_{ir}(x) = \int\limits_{\nu_y}\int\limits_0^T u(y, \varphi) G_{ir}(y, \varphi, x, T)\, dy\, d\varphi \quad (i = 1, 2, \ldots, n;\ r = 0, 1, \ldots, p) \tag{2.9}$$

it is not difficult to find that

$$u(z, \tau) = \int\limits_{\nu_x} \sum_{i=1}^n \Psi_i(x) \Big\{\mu_i g_{i0}(x) G_{i0}(z, \tau, x, T) +$$

$$+ \sum_{r=1}^p \sum_{k=1}^n [\theta_r \lambda_{ik} g_{kr}(x) G_{ir}(z, \tau, x, T)] + \frac{1}{2} \eta_i G_{i0}(z, \tau, x, T)\Big\}\, dx \tag{2.10}$$

If Eq. (2.10) is substituted into (2.9), we obtain the following system of integral equations:

$$g_{sj}(\zeta) = \int_{v_x} \sum_{i=1}^{n} \Psi_i(x) \Big\{ \mu_i g_{i0}(x) \varkappa_{sji0}(\zeta, x) + $$

$$+ \sum_{r=1}^{p} \sum_{k=1}^{n} [\theta_r \lambda_{ik} g_{kr}(x) \varkappa_{sjir}(\zeta, x)] + \frac{1}{2} \eta_i \varkappa_{sji0}(\zeta, x) \Big\} dx \tag{2.11}$$

$$(s = 1, 2, \ldots, n;\ j = 0, 1, 2, \ldots, p)$$

Here

$$\varkappa_{sjir}(\zeta, x) = \int_{v_z} \int_0^T G_{sj}(z, \tau, \zeta, T) G_{ir}(z, \tau, x, T)\, dz\, d\tau \tag{2.12}$$

$$(i, s = 1, 2, \ldots, n;\ j, r = 0, 1, 2, \ldots, p)$$

With due regard to notation (2.9), equality (1.8) can be rewritten thus:

$$\gamma_i(x, T) = \eta_i g_{i0}(x) + \mu_i g_{i0}^2(x) + \sum_{r=1}^{n} \sum_{k=1}^{n} [\theta_r \lambda_{ik} g_{ir}(x) g_{kr}(x)] \qquad (i = 1, 2, \ldots, n) \tag{2.13}$$

From the system of $n(p+2)$ Eqs. (2.11), (2.13) we must find the $n(p+2)$ unknown functions $g_{sj}(x)$, $\Psi_i(x)$ $(i, s = 1, 2, \ldots, n;\ j = 0, 1, 2, \ldots, p)$. The solution of the system of Eqs. (2.11), (2.13) is not unique as a consequence of the nonlinearity of system (2.13). From all these solutions it is necessary to choose that which furnishes the minimum of the control norm (2.8). Using relation (2.9) we can rewrite relation (2.8) thus:

$$\|u\|^2 = \int_{v_x} \sum_{i=1}^{n} \Psi_i(x) \Big[\gamma_i(x, T) - \frac{1}{2} g_{i0}(x)\Big] dx \tag{2.14}$$

We summarize the results stated above in the following way.

Theorem 1. The solution algorithm for the auxiliary problem consists of the following procedure: (a) from the system of $n(p+2)$ of Eqs. (2.11), (2.13) we find the $n(p+2)$ unknown functions $g_{sj}(x)$, $\Psi_i(x)$ $(s, i = 1, 2, \ldots, n;\ j = 0, 1, 2, \ldots, p)$, (b) if the solution of this system is unique, then the optimal control $u(z, \tau)$ and its norm $\|u\|$ are found by formulas (2.10) and (2.14), respectively, (c) if the solution of system (2.11), (2.13) is not unique, then from all these solutions we select that one which yields the minimum of norm (2.14).

Note 2. From formula (2.14) it follows that for the norm of the optimal control to be continuous in the parameter T it is sufficient that the given functions $\gamma_i(x, T)$ $(i = 1, 2, \ldots, n)$ and the solution of the system of Eqs. (2.11), (2.13) be continuous in T.

3. Solution algorithm for Problem 1. We state the following theorem.

Theorem 2. Let $u'(z, \tau)$ be the optimal control for the auxiliary problem. If for this problem the norm $\|u\|$ depends continuously on T, then the smallest positive root T' of the equation

$$\|u'\|^2 \equiv \int_{v_x} \sum_{i=1}^{n} \Psi_i(x) \Big[\gamma_i(x, T) - \frac{1}{2} g_{i0}(x)\Big] dx = a^2 \tag{3.1}$$

yields the time-optimal time for Problem 1.

Proof of Theorem 2. If $u(z, \tau) \in L_2[v_z \times (0 \leqslant \tau \leqslant T)]$ is any control satisfying system (1.8), then

$$\| u' \| = \min \| u \| \leqslant \| u \| \tag{3.2}$$

Note that

$$\| u' \| \to \infty \quad \text{as} \quad T \to 0 \tag{3.3}$$

Indeed, this fact follows from the inequalities

$$0 < \int\limits_{\nu_x} \sum_{i=1}^{n} \gamma_i^2 (x, T)\, dx \leqslant$$

$$\leqslant 2 \| u \|^2 \int\limits_{\nu_z} \int\limits_0^T \int\limits_{\nu_x} \sum_{i=1}^{n} [\eta_i G_{i0} (z, \tau, x, T)]^2 \, dz\, d\tau\, dx +$$

$$+ 2 \| u \|^4 \int\limits_{\nu_z} \int\limits_0^T \int\limits_{\nu_y} \int\limits_0^T \int\limits_{\nu_x} \sum_{i=1}^{n} H_i^2 (z, \tau, y, \varphi, x, T)\, dz\, d\tau\, dy\, d\varphi\, dx \tag{3.4}$$

which ensue from (1. 8).

Let us assume that T_1 is the time-optimal time for Problem 1. Further, let the norm

$$\| u' \| |_{T_1} < a \tag{3.5}$$

i. e. equality (3. 1) is not fulfilled. Since the norm $\| u' \|$ depends continuously on T and satisfies condition (3. 3), there exists a time $T_2 < T_1$ such that the inequalities

$$\| u' \| |_{T_1} < \| u' \| |_{T_2} \leqslant a \tag{3.6}$$

are fulfilled. Consequently, time T_1 is not the least control time. Since the norm of optimal control (2. 7) $\| u' \| \leqslant \| u \|$, where $u (z, \tau) \in L_2 [\nu_z \times (0 \leqslant \tau \leqslant T)]$ is some solution of system (1. 8), and since by virtue of condition (3. 3) the norm $\| u \| > a$ for $T < T'$, the smallest root T' of Eq. (3. 1) is the time-optimal time for Problem 1. Thus, Theorem 2 is completely proved.

Note 3. If as a function of T the norm $\| u' \|$ has discontinuities of the first kind, then equality (3. 1) may not necessarily be fulfilled in the time-optimal case. In truth, this does not mean that the approach described above is inapplicable in this case also for solving the time-optimal problem. We remark that here it suffices to find the least value of T for which $\min \| u \| \leqslant a$.

Note 4. If the auxiliary problem has a solution $u (z, \tau) \in L_2 [\nu_z \times (0 \leqslant \tau \leqslant T)]$ for any value of parameter $T \in (0, \infty)$ and if

$$\| u' \| \to 0 \quad \text{as} \quad T \to \infty \tag{3.7}$$

then the solution of Problem 1 exists for any value of the constant $a \in (0, \infty)$, i. e. we can always find a T for which the condition $\| u \| \leqslant a$ is fulfilled. Note that according to relation (2. 8), for the fulfillment of condition (2. 7) it is necessary that

$$\| \Psi (x) \|^2 \equiv \int\limits_{\nu_x} \sum_{i=1}^{n} \Psi_i^2 (x)\, dx \to 0 \quad \text{as} \quad T \to \infty \tag{3.8}$$

4. Formulation of Problem 2. In aeroelastic problem it is usually assumed that wing design is ideally elastic [1, 2]. Under the assumption rotary motions and torsional oscillations of a "flying wing" can be described by the partial differential equation

$$\alpha (x)\, Q_{tt} - \frac{\partial}{\partial x} (\delta (x)\, Q_x) = \mu (x, t) + F_1 (x)\, F_2 (t)\, u (x, t) \tag{4.1}$$

Here $Q = Q(x, t)$ is the angle between the chords of section x (at an instant t) in unperturbed and perturbed flights, $\alpha(x)$ is the lineal mass moment of inertia, $\delta(x)$ is the rigidity of the wing section to torsion, $\mu(x, t)$ is the supplemental lineal torque which arises in perturbed flight, $u(x, t)$ is the deterministic control. The product $F_1(x)F_2(t)u(x, t)$ has the sense of a distributed torque, where the function $F_1(x)$ characterizes the distribution of the control along the wing length, while the influence on the control of various random factors (for example, atmospheric turbulence) is taken into account by means of the function $F_2(t)$.

For Eq. (4.1) the initial and boundary conditions are

$$Q(x, 0) = f_1(x), \qquad Q_t(x, t)|_{t=0} = f_2(x) \tag{4.2}$$

$$Q_x(x, t)|_{x=0} = 0, \qquad Q_x(x, t)|_{x=l} = 0 \tag{4.3}$$

Here l is the wing span, $f_1(x)$, $f_2(x)$ are functions depending on a random parameter and taking their own distributions with a specified probability from a certain set of function-realizations, the realizations $f_1(x)$ are continuous piecewise-differentiable functions, the realizations $f_2(x)$ are piecewise-continuous functions.

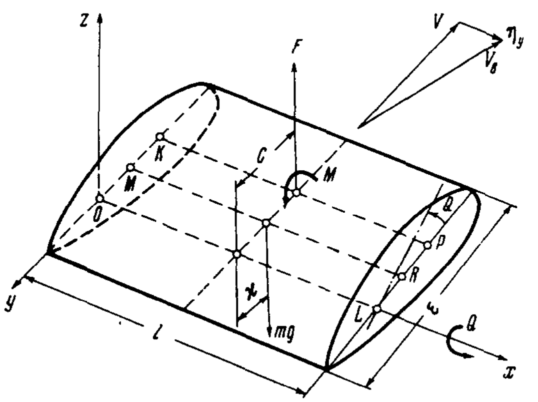

Fig. 1

We assume that: (a) the "flying wing" has a rectangular form in plan, (b) during the unperturbed motion the wing is in level flight in a calm atmosphere, (c) the motion of the wing is considered to start at the instant $t = 0$ and, at this instant the atmosphere becomes turbulent, (d) the flight altitude and velocity remain unchanged, (e) the functions $\alpha(x)$, $\delta(x)$ are taken to be independent of the space variable x.

With the observance of assumption (b) the motion of each wing section is in steady-state and takes place at an unchanged angle of attack $\omega(x)$ which is measured from the direction of zero lift. This angle of attack is the sum of the local angle of attack $\omega_0(x)$ measured from the direction of zero lift without taking into account the elastic twisting angle $q_0(x)$ in the steady state level flight and of the angle $q_0(x)$ itself, i.e. $\omega(x) = \omega_0(x) + q_0(x)$. Here the lineal torque does not depend on time t and, under certain commonly-accepted assumptions, can be described in the following manner [1, 2]:

$$\mu'(x) = \varepsilon c d S \omega(x) + M - mg\varkappa \tag{4.4}$$

$$M = \varepsilon^2 S E_\mu, \quad d = \left.\frac{\partial E_y}{\partial \omega}\right|_{\omega=0}, \quad S = \frac{\rho V^2}{2}, \quad F = \varepsilon S E_y$$

Here ε is the wing chord, c is the distance between the elastic axis (the line OL) and the line KP which passes through the aerodynamic center of the wing profile at the given section, E_μ is moment coefficient relative to line KP, E_y is the lift coefficient of the wing section, ρ is the density of the free-stream flow, V is the flight velocity, $\varkappa$ is the distance between points of the lines OL and MR at the given wing section, MR is the mass center line of the wing section (see Fig. 1).

For determining the aerodynamic forces and moments in the nonsteady-state motion we make use of the lifting-strip theory [1, 2]. The equality

$$\mu'(x) + \mu(x, t) = \varepsilon c d S\,[\omega(x) + Q(x, t) + \omega_\eta(x, t)] + M - \\ - mg\varkappa - \Gamma(x) Q(x, t) \tag{4.5}$$

holds in this case. Here $\Gamma(x)$ is the elastic wing stiffness, ω_η is the angle of attack caused by a change in the velocity vector by the atmospheric turbulence. It is commonly accepted that the perturbed velocity $V_p \approx V$. It is obvious that

$$\omega_\eta = \operatorname{arctg}[\eta_y / V] \approx \eta_y / V \tag{4.6}$$

From formulas (4.4), (4.5) it follows that

$$\mu(x, t) = \varepsilon c d S\,[Q(x, t) + \omega_\eta(x, t)] - \Gamma(x) Q(x, t) \tag{4.7}$$

It is known [8] that the mean $\langle \eta_y \rangle = 0$ for homogeneous and isotropic turbulence. Taking relation (4.7) into account, Eq. (4.1) can be rewritten as follows:

$$Q_{tt} = b^2 Q_{xx} - AQ + B\eta_y + \frac{1}{\alpha} F_1(x) F_2(t) u(x, t) \tag{4.8}$$

Here

$$A = \frac{1}{\alpha}[\Gamma(x) - dc\varepsilon S], \quad b^2 = \frac{\delta}{\alpha} \equiv \text{const}, \quad B = \frac{\varepsilon c\, dS}{\alpha V} \tag{4.9}$$

The initial and boundary conditions for Eq. (4.8) remain as before (see formulas (4.2) and (4.3), respectively). We take it that the means have been given for the random variables η_y, $F_2(t)$, $f_1(x)$, $f_2(x)$.

During the flight the three situations listed below are possible, depending on the velocity: (a) if $A > 0$, stability holds for the wing with respect to torsional strain and angular motion, (b) if $A < 0$, the wing becomes unstable with respect to torsional strain and angular motion, (c) if $A = 0$, the wing is found to be in a state which is often called critical. Let us assume that $\Gamma(x)$ and d do not depend on x. Below we shall consider only the case when $A \geqslant 0$.

For each realization of the random functions the solution of the boundary value Problem 2, described by the collection of formulas (4.8), (4.2), (4.3), can be characterized by the following formulas [9]:

$$Q(x, t) = \theta(x, t) + \sum_{k=1}^{\infty} \Big\{ a_k \cos(\lambda_k t) + b_k \sin(\lambda_k t) + \frac{1}{\lambda_k} \int_0^l \int_0^t V_k(\zeta) \times \\ \times \sin[\lambda_k (t - \tau)] \Big[B\eta_y + \frac{1}{\alpha} F_1(\zeta) F_2(\tau) u(\zeta, \tau) \Big] d\zeta\, d\tau \Big\} V_k(x) \tag{4.10}$$

Here, if $A > 0$, then

$$\theta(x, t) = \Big\{ \frac{1}{\sqrt{A}} \int_0^l \int_0^t V_0(\zeta) \Big[B\eta_y + \frac{1}{\alpha} F_1(\zeta) F_2(\tau) u(\zeta, \tau) \Big] \sin[\sqrt{A}(t - \tau)] \times \\ \times d\zeta\, d\tau + a_0 \cos(\sqrt{A} t) + b_0 \sin(\sqrt{A} t) \Big\} V_0(x) \tag{4.11}$$

$$\omega_k = \frac{k\pi}{l}, \quad \lambda_k{}^2 = (A + b^2 \omega_k{}^2) \qquad (k = 0, 1, 2, \ldots) \tag{4.12}$$

$$V_0(x) = \frac{1}{\sqrt{l}}; \quad V_k(x) = \sqrt{\frac{2}{l}} \cos(\omega_k x) \qquad (k = 1, 2, \ldots) \tag{4.13}$$

$$a_0 = \int_0^l f_1(x) V_0(x)\, dx, \qquad b_0 = \frac{1}{\sqrt{A}} \int_0^l f_2(x) V_0(x)\, dx \tag{4.14}$$

$$a_k = \int_0^l f_1(x) V_k(x)\, dx, \quad b_k = \frac{1}{\lambda_k} \int_0^l f_2(x) V_k(x)\, dx \qquad (k = 1, 2, \ldots) \tag{4.15}$$

However, if $A = 0$, expressions (4.11), (4.14) transform to

$$\theta(x, t) = \Big\{a_0 + b_0' t + \int_0^l \int_0^t (t - \tau) V_0(\zeta) \times$$
$$\times \Big[B\eta_v + \frac{1}{\alpha} F_1(\zeta) F_2(\tau) u(\zeta, \tau)\Big] d\zeta\, d\tau\Big\} V_0(x) \tag{4.16}$$

$$b_0' = \int_0^l f_2(x) V_0(x)\, dx \tag{4.17}$$

The deterministic control $u\ (x,\ t)$ will be called admissible if the conditions

$$u(x, t) \in L_2, \quad \|u\| \equiv \Big(\int_0^l \int_0^T u^2(x, t)\, dx\, dt\Big)^{1/2} \leqslant a \tag{4.18}$$

where a is a given positive number, are fulfilled.

We pose the following problem: find an admissible control $u\ (x,\ t)$ which in least possible time T would take system (4.8), (4.3) from the initial state (4.2) into the state described by the formulas

$$\langle Q(x, t)\rangle|_{t=T} = 0, \qquad \langle Q_t(x, t)\rangle|_{t=T} = 0 \tag{4.19}$$

5. Solution algorithm for Problem 2. If we compute the means $\langle Q\ (x,\ t)\rangle$, $\langle Q_t\ (x,\ t)\rangle$, then, using conditions (4.19) and $\langle\eta_v\rangle = 0$, after a number of transformations, we can obtain the following relations

$$\gamma_i(x, T) = \int_0^l \int_0^T u(\zeta, \tau) G_i(\zeta, \tau, x, t)\, d\zeta\, d\tau \qquad (i = 1, 2) \tag{5.1}$$

Here,

$$\gamma_i(x, T) = \sum_{k=0}^{\infty} [\gamma_{ik} V_k(x)] \qquad (i = 1, 2) \tag{5.2}$$

$$\gamma_{10} = -(h_1 + \rho_1), \quad \gamma_{1k} = -a_k' \cos(\lambda_k T) - b_k' \sin(\lambda_k T) \qquad (k = 1, 2, \ldots) \tag{5.3}$$

$$\gamma_{20} = -(h_2 + \rho_2), \quad \gamma_{2k} = \lambda_k [a_k' \sin(\lambda_k T) - b_k' \cos(\lambda_k T)] \qquad (k = 1, 2, \ldots) \tag{5.4}$$

$$a_k' = \int_0^l V_k(x) \langle f_1(x)\rangle\, dx, \quad b_k' = \frac{1}{\lambda_k} \int_0^l V_k(x) \langle f_2(x)\rangle\, dx \qquad (k = 0, 1, 2, \ldots) \tag{5.5}$$

$$b_0^* = \int_0^l V_0(x) \langle f_2(x) \rangle dx \tag{5.6}$$

$$G_1(\zeta, \tau, x, T) = \frac{1}{\alpha} \sum_{k=1}^{\infty} \left\{ \frac{1}{\lambda_k} \sin [\lambda_k (T-\tau)] F_1(\zeta) \times \right.$$
$$\left. \times \langle F_2(\tau) \rangle V_k(x) V_k(\zeta) \right\} + \Phi_1(\zeta, \tau, x, T) \tag{5.7}$$

$$G_2(\zeta, \tau, x, T) = \frac{1}{\alpha} \sum_{k=1}^{\infty} \{\cos [\lambda_k (T-\tau)] F_1(\zeta) \langle F_2(\tau) \rangle V_k(x) V_k(\zeta)\} +$$
$$+ \Phi_2(\zeta, \tau, x, T) \tag{5.8}$$

moreover, if $A > 0$, then

$$\Phi_1(\zeta, \tau, x, T) = \frac{1}{\alpha \sqrt{A}} F_1(\zeta) \langle F_2(\tau) \rangle V_0(\zeta) V_0(x) \sin \left[\sqrt{A}(T-\tau)\right]$$

$$\Phi_2(\zeta, \tau, x, T) = \frac{1}{\alpha} F_1(\zeta) \langle F_2(\tau) \rangle V_0(\zeta) V_0(x) \cos \left[\sqrt{A}(T-\tau)\right]$$
$$h_1 = a_0' \cos\left(\sqrt{A}T\right), \qquad \rho_1 = b_0' \sin\left(\sqrt{A}T\right) \tag{5.9}$$

$$h_2 = -a_0' \sqrt{A} \sin\left(\sqrt{A}T\right), \qquad \rho_2 = b_0' \sqrt{A} \cos\left(\sqrt{A}T\right) \tag{5.10}$$

if, however, $A = 0$, then

$$\Phi_1(\zeta, \tau, x, T) = \frac{1}{\alpha} F_1(\zeta) \langle F_2(\tau) \rangle (T-\tau) V_0(\zeta) V_0(x)$$

$$\Phi_2(\zeta, \tau, x, T) = \frac{1}{\alpha} F_1(\zeta) \langle F_2(\tau) \rangle V_0(\zeta) V_0(x) \tag{5.11}$$

$$h_1 = a_0', \quad \rho_1 = b_0^* T, \qquad h_2 = 0, \qquad \rho_2 = 0$$

Thus, the solution of Problem 2 reduces to the investigation of system (5.1) which was studied in Sects. 1–3 of this paper. In accordance with these results the optimal control has the form

$$U(\zeta, \tau) = \int_0^l \sum_{i=1}^{2} \Psi_i(x) G_i(\zeta, \tau, x, T) dx \tag{5.12}$$

Using the expansion

$$\Psi_i(x) = \sum_{k=0}^{\infty} [\Psi_{ik} V_k(x)] \qquad (i = 1, 2) \tag{5.13}$$

optimal control (5.12) can be represented in the following manner:

$$u(\zeta, \tau) = \frac{1}{\alpha} \Xi(\zeta, \tau) + \frac{1}{\alpha} \sum_{k=1}^{\infty} \left[\frac{1}{\lambda_k} \Psi_{1k} \sin [\lambda_k (T-\tau)] + \right.$$
$$\left. + \Psi_{2k} \cos [\lambda_k (T-\tau)] \right] F_1(\zeta) \langle F_2(\tau) \rangle V_k(\zeta) \tag{5.14}$$

Here,

$$\Xi(\zeta, \tau) = F_1(\zeta) \langle F_2(\tau) \rangle V_0(\zeta) \left\{ \frac{1}{\sqrt{A}} \Psi_{10} \sin [\sqrt{A}(T-\tau)] + \right.$$
$$\left. + \Psi_{20} \cos [\sqrt{A}(T-\tau)] \right\} \qquad \text{for} \quad A > 0 \tag{5.15}$$

$$\Xi(\zeta, \tau) = F_1(\zeta)\langle F_2(\tau)\rangle V_0(\zeta)\{(T-\tau)\Psi_{10} + \Psi_{20}\} \quad \text{for} \quad A = 0 \quad (5.16)$$

The smallest root T of the equation

$$\|u\|^2 = \frac{1}{\alpha^2}\int_0^l\int_0^T \left\{\sum_{k=1}^{\infty}\left[\frac{1}{\lambda_k}\Psi_{1k}\sin[\lambda_k(T-\tau)] + \Psi_{2k}\cos[\lambda_k(T-\tau)]\right] \times\right.$$

$$\left.\times F_1(\zeta)\langle F_2(\tau)\rangle V_k(\zeta) + \Xi(\zeta, \tau)\right\}^2 d\zeta\, d\tau \equiv \sum_{k=0}^{\infty}\sum_{i=1}^{2}(\Psi_{ik}\gamma_{ik}) = a^2 \quad (5.17)$$

yields the time-optimal time. The unknown constants Ψ_{1k}, Ψ_{2k} $(k = 0, 1, 2, \ldots)$ are found from the system of equations presented above, which can be obtained if in system (5.1) we substitute the value of $u(\zeta, \tau)$ described by formula (5.14). In summary we obtain that

$$\sum_{i=0}^{\infty}\{E_{ik}(A_{ik}\Psi_{1i} + B_{ik}\Psi_{2i})\} = \gamma_{1k} \quad (k = 0, 1, 2, \ldots) \quad (5.18)$$

$$\sum_{i=0}^{\infty}\{E_{ik}(C_{ik}\Psi_{1i} + D_{ik}\Psi_{2i})\} = \gamma_{2k} \quad (k = 0, 1, 2, \ldots) \quad (5.19)$$

Here,

$$A_{ik} = A_{ki} = \frac{1}{\lambda_k\lambda_i}\int_0^T \langle F_2(\tau)\rangle^2 \sin[\lambda_k(T-\tau)]\sin[\lambda_i(T-\tau)]\,d\tau$$

$$B_{ik} = C_{ki} = \frac{1}{\lambda_k}\int_0^T \langle F_2(\tau)\rangle^2 \sin[\lambda_k(T-\tau)]\cos[\lambda_i(T-\tau)]\,d\tau$$

$$D_{ik} = D_{ki} = \int_0^T \langle F_2(\tau)\rangle^2 \cos[\lambda_k(T-\tau)]\cos[\lambda_i(T-\tau)]\,d\tau$$

$$E_{ik} = E_{ki} = \frac{1}{\alpha^2}\int_0^l F_1^2(\zeta)V_i(\zeta)V_k(\zeta)\,d\zeta \qquad (i, k = 0, 1, 2, \ldots)$$

If, however, $A = 0$, the coefficients A_{0k}, B_{0k}, C_{0k}, D_{0k} $(k = 0, 1, 2, \ldots)$ have a somewhat different form, namely,

$$A_{00} = \int_0^T \langle F_2(\tau)\rangle^2 (T-\tau)^2\,d\tau, \qquad D_{00} = \int_0^T \langle F_2(\tau)\rangle^2\,d\tau$$

$$B_{00} = C_{00} = \int_0^T (T-\tau)\langle F_2(\tau)\rangle^2\,d\tau$$

$$A_{i0} = A_{i0} = \frac{1}{\lambda_i}\int_0^T (T-\tau)\langle F_2(\tau)\rangle^2 \sin[\lambda_i(T-\tau)]\,d\tau$$

$$B_{i0} = \int_0^T (T-\tau)\langle F_2(\tau)\rangle^2 \cos[\lambda_i(T-\tau)]\,d\tau$$

$$C_{i0} = \frac{1}{\lambda_i}\int_0^T \langle F_2(\tau)\rangle^2 \sin[\lambda_i(T-\tau)]\,d\tau$$

$$D_{0i}=D_{i0}=\int_0^T \langle F_2(\tau)\rangle^2 \cos[\lambda_i(T-\tau)]\,d\tau \qquad (i=1,2,\ldots)$$

Methods for solving system (5.18), (5.19) (for a fixed value of T) can be found, for example, in [10].

Note 5. From formulas (5.14)–(5.19) it follows that the solution of Problem 2 depends essentially on the form of the function $F_1(x)$. The formulas presented above in Sect. 5 simplify considerably, for example, in cases such as:

1) the control is distributed over the whole wing axis, i.e.

$$F_1(x)=1 \qquad (0\leqslant x\leqslant l)$$

2) the control is applied to two parts of the wing axis, symmetric relative to the middle, i.e.

$$F_1(x)=\begin{cases}1, & \text{if } a\leqslant x\leqslant b,\quad l-b\leqslant x\leqslant l-a\\ 0, & \text{if } 0\leqslant x<a,\quad b<x<l-b,\quad l-a<x\leqslant l\end{cases}$$

3) the control is applied to the central part of the wing axis, i.e.

$$F_1(x)=\begin{cases}1, & \text{if } a\leqslant x\leqslant l-a\\ 0, & \text{if } 0\leqslant x<a,\quad l-a<x\leqslant l\end{cases}$$

4) the form of the function $F_1(x)$ along the whole wing axis is preassigned and when the optimal control is sought in the form of a function depending only on t.

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